A TYPE OF DIFFERENTIAL SYSTEM CONTAINING A PARAMETER*

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Poincaré† and others have discussed the continuity with respect to a parameter μ of the solutions of a system of equations

$$\frac{dx_i}{dt} = X_i(x_1, x_2, \ldots, x_n, \mu, t),$$

in which the function X_i is analytic in μ if $|\mu| < c$, and satisfies certain other conditions with respect to $x_1, x_2, \ldots, x_n, t$ in a domain

$$|x_i - x_i^0| < b, \qquad 0 \leqq t \leqq T \qquad (i = 1, 2, \ldots, n).$$

For certain problems in mechanics it is convenient to have a similar discussion of equations of the form

$$\frac{dx_i}{dt} = X_i(x_1, x_2, \ldots, x_n, \cos\nu t, \sin\nu t) \qquad (i = 1, 2, \ldots, n)$$

for very large values of the parameter ν; the present paper is devoted to this type of equations, and an application is made to a problem related to the restricted problem of three bodies.

1. Integration by the method of successive approximations

In the equations

$$(1) \qquad \frac{dx_i}{dt} = X_i(x_1, x_2, \ldots, x_n, \cos\nu t, \sin\nu t)$$

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† *Les Méthodes nouvelles de la Mécanique céleste,* vol. I, § 27.

assume that the functions X_i are continuous in all their arguments, and satisfy the conditions

$$
\begin{aligned}
&|X_i(x_1'', \ldots, x_n'', \cos\nu t, \sin\nu t) - X_i(x_1', \ldots, x_n', \cos\nu t, \sin\nu t)| \\
(2) \qquad &< A_1|x_1'' - x_1'| + \cdots + A_n|x_n'' - x_n'|, \\
&|X_i| < M \qquad (i = 1, 2, \ldots. n),
\end{aligned}
$$

if the arguments lie in the domain

$$(D) \qquad |x_i - x_i^0| < b, \quad -\infty < t < \infty \quad (i = 1, 2, \ldots, n).$$

Then if $K = \sum_{i=1}^{n} A_i$, equations (1) can be integrated by the method of successive approximations,* and the solutions are defined if

$$(3) \qquad 0 \leqq t - t_0 < \frac{1}{K} \log\left(1 + \frac{bK}{M}\right).$$

In the discussion of the successive approximations we shall require the following

LEMMA. *Assume that* $x_1(t), \ldots, x_n(t)$ *are continuous, possess continuous derivatives, and that* $|x_i - x_i^0| < b$ *for* $0 \leqq t \leqq T$, *and suppose* $y_i(t)$, $Y_i(t)$ *defined by the equations*

$$
\begin{aligned}
&Y_i(t) = \int_0^t X_i(x_1, \ldots, x_n, \cos\nu t, \sin\nu t)\, dt, \\
(4) \qquad & \\
&y_i(t) = \int_0^t \frac{1}{2\pi} \int_0^{2\pi} X_i(x_1, \ldots, x_n, \cos\omega, \sin\omega)\, d\omega\, dt.
\end{aligned}
$$

* Picard, *Traité d' Analyse*, vol. 2, 1905, Chap. XI.

Then if ϵ is an arbitrarily small positive quantity, N can be found such that if $\nu > N$,

$$|Y_i(t) - y_i(t)| < \epsilon, \qquad 0 \leqq t \leqq T.$$

If $\Delta t = \frac{2\pi}{\nu}$,

$$\cos\nu(t + \Delta t) = \cos\nu t, \quad \sin\nu(t + \Delta t) = \sin\nu t.$$

Suppose

$$m\Delta t \leqq t' < (m+1)\Delta t, \qquad m\Delta t = \bar{t}.$$

Then

$$(5) \quad \begin{aligned} Y_i(t') = \sum_{k=1}^{m} \int_{(k-1)\Delta t}^{k\Delta t} X_i(x_1, x_2, \ldots, x_n, \cos\nu t, \sin\nu t)\, dt \\ + \int_{\bar{t}}^{t'} X_i(x_1, x_2, \ldots, x_n, \cos\nu t, \sin\nu t)\, dt. \end{aligned}$$

Since $\nu(k-1)\Delta t = 2(k-1)\pi$, $\nu k\Delta t = 2k\pi$,

$$I_k = \int_{(k-1)\Delta t}^{k\Delta t} X_i(x_1, \ldots, x_n, \cos\nu t, \sin\nu t)\, dt$$

$$= \frac{1}{\nu}\int_0^{2\pi} X_i(x_1, \ldots, x_n, \cos\omega, \sin\omega)\, d\omega$$

if

$$\omega = \nu[t - (k-1)\Delta t],$$

or finally

$$(6) \qquad I_k = \frac{\Delta t}{2\pi}\int_0^{2\pi} X_i(x_1^{(k-1)}, \ldots, x_n^{(k-1)}, \cos\omega, \sin\omega)\, d\omega,$$

where $x_i^{(k-1)}$ denotes the function $x_i(t)$ in the interval $(k-1)\Delta t \leqq t \leqq k\Delta t$, or the corresponding function of ω. If $\bar{x}_i^{(k-1)} = x_i[(k-1)\Delta t]$, we obtain from (2)

$$|X_i(x_1^{(k-1)}, \ldots, x_n^{(k-1)}, \cos\omega, \sin\omega) - X_i(\bar{x}_1^{(k-1)}, \ldots, \bar{x}_n^{(k-1)}, \cos\omega, \sin\omega)|$$

$$< \sum_{r=1}^{n} A_r \left| x_r^{(k-1)} - \bar{x}_r^{(k-1)} \right|.$$

Since $x_i'(t)$ exists and is continuous for $0 \leqq t \leqq T$, we have, for some constant B,

$$|x_i'(t)| < B, \qquad 0 \leqq t \leqq T.$$

Then

$$\left| x_i^{(k-1)} - \bar{x}_i^{(k-1)} \right| < B\Delta t,$$

(7) $$\sum_{i=1}^{n} A_i \left| x_i^{(k-1)} - \bar{x}_i^{(k-1)} \right| < BK\Delta t.$$

If

$$I_k' = \frac{\Delta t}{2\pi} \int_0^{2\pi} X_i(\bar{x}_1^{(k-1)}, \ldots, \bar{x}_n^{(k-1)}, \cos\omega, \sin\omega)\, d\omega,$$

$$|I_k - I_k'| \leqq \frac{\Delta t}{2\pi} \int_0^{2\pi} BK\Delta t\, d\omega$$

(8) $$\leqq BK(\Delta t)^2.$$

Also

(9) $$\left| \int_{\bar{t}}^{t'} X_i(x_1, \ldots, x_n, \cos\nu t, \sin\nu t)\, dt \right| < M\Delta t.$$

From (5), (8), (9),

$$\left| Y_i(t') - \sum_{k=1}^{m} I'_k \right| \leqq M\Delta t + BKm(\Delta t)^2$$

$$\leqq \Delta t (M + BK\bar{t}). \tag{10}$$

From the definition of $y_i(t)$,

$$y_i(\bar{t}) = \int_0^{\bar{t}} \frac{1}{2\pi} \int_0^{2\pi} X_i(x_1, x_2, \ldots, x_n, \cos\omega, \sin\omega)\, d\omega\, dt$$

$$= \sum_{k=1}^{m} \frac{\Delta t}{2\pi} \int_0^{2\pi} X_i\,[x_1^{(k-1)}(\xi), x_2^{(k-1)}(\xi), \ldots, x_n^{(k-1)}(\xi), \cos\omega, \sin\omega]\, d\omega,$$

$$(k-1)\,\Delta t \leqq \xi \leqq k\Delta t,$$

from the theorem of the mean. Also,

$$|\, y_i(\bar{t}) - v_i(t')\,| < M\Delta t.$$

Employing inequalities (2),

$$\left| y_i(\bar{t}) - \sum_{k=1}^{m} I'_k \right| \leqq BK\bar{t}\Delta t.$$

Hence, finally,

$$\left| y_i(t') - \sum_{k=1}^{m} I'_k \right| \leqq \Delta t\,[M + BK\bar{t}]. \tag{11}$$

Combining (10) and (11),

$$(12) \qquad |Y_i(t') - y_i(t')| \leqq \frac{4\pi}{\nu}[M + BKT], \quad 0 \leqq t' \leqq T.$$

From (12) the lemma follows immediately.

Equations (1) can be integrated by the construction of the functions

$$(13) \quad u_i^{(k)} = x_i^0 + \int_0^t X_i(u_1^{(k-1)}, u_2^{(k-1)}, \ldots, u_n^{(k-1)}, \cos\nu t, \sin\nu t)\,dt,\ u_i^0 = x_i^0$$

$$(i = 1, 2, \ldots, n;\ k = 1, 2, \ldots).$$

Consider the differential equations

$$(14) \quad \frac{dz_i}{dt} = \frac{1}{2\pi}\int_0^{2\pi} X_i(z_1, z_2, \ldots, z_n, \cos\omega, \sin\omega)\,d\omega \quad (i = 1, 2, \ldots, n),$$

and suppose these integrated by the same method;

$$(15) \quad v_i^{(k)} = x_i^0 + \int_0^t \frac{1}{2\pi}\int_0^{2\pi} X_i(v_1^{(k-1)}, v_2^{(k-1)}, \ldots, v_n^{(k-1)}, \cos\omega, \sin\omega)\,d\omega\,dt,$$

$$v_i^0 = x_i^0, \qquad (i = 1, 2, \ldots, n;\ k = 1, 2, \ldots).$$

It is seen immediately that the same constants K, M can be employed for equations (14). Also the functions $u_i^{(k)}$, $v_i^{(k)}$ lie in the domain D.

From (13)

$$u_i^{(k)} = x_i^0 + \int_0^t X_i(v_1^{(k-1)}, v_2^{(k-1)}, \ldots, v_n^{(k-1)}, \cos\nu t, \sin\nu t)\,dt$$

$$(16) \qquad + \int_0^t [X_i(u_1^{(k-1)}, \ldots, u_n^{(k-1)}, \cos\nu t, \sin\nu t)$$

$$- X_i(v_1^{(k-1)}, \ldots, v_n^{(k-1)}, \cos\nu t, \sin\nu t)]\,dt.$$

Suppose $|u_i^{(k-1)} - v_i^{(k-1)}| < \varrho$ $(i = 1, 2, \ldots, n)$; then if $w_i^{(k)}$ is defined by the equation

$$w_i^{(k)} = x_i^0 + \int_0^t X_i(v_1^{(k-1)}, \ldots, v_n^{(k-1)}, \cos\nu t, \sin\nu t)\, dt,$$

$$|u_i^{(k)} - w_i^{(k)}| < K\varrho t \qquad (i = 1, 2, \ldots, n).$$

Also, from the lemma,

$$|w_i^{(k)} - v_i^{(k)}| \leqq \frac{4\pi}{\nu}[M + BKT].$$

Hence

$$|u_i^{(k)} - v_i^{(k)}| \leqq K\varrho t + \frac{4\pi}{\nu}[M + BKT].$$

From the definition of the functions u, v, w it follows that B can be replaced by M.

If $k = 1$, we obtain

$$|u_i' - v_i'| \leqq \frac{4\pi M}{\nu}[1 + KT] = \frac{C_1}{\nu}.$$

Similarly,

$$|u_i'' - v_i''| \leqq \frac{KTC_1}{\nu} + \frac{C_1}{\nu} = \frac{C_2}{\nu},$$

and in general

$$(17) \qquad |u_i^{(k)} - v_i^{(k)}| \leqq \frac{C_k}{\nu} \qquad (k = 1, 2, \ldots),$$

where

$$C_k = 4\pi M(1 + KT)\sum_{r=0}^{k-1}(KT)^r.$$

Now if $T < \frac{1}{K} \log \left(1 + \frac{bK}{M}\right)$, the integer ι can be chosen so large that if $\sigma > \iota$, $0 < t \leqq T$,

$$|x_i(t) - u_i^{(\sigma)}(t)| < \frac{\varepsilon}{3},$$

$$|z_i(t) - v_i^{(\sigma)}(t)| < \frac{\varepsilon}{3},$$

if ε is any previously assigned positive quantity.

Now suppose σ fixed; from (17)

$$|u_i^{(\sigma)} - v_i^{(\sigma)}| \leqq \frac{C_\sigma}{\nu}.$$

Hence if ν is chosen sufficiently large $\frac{C_\sigma}{\nu} < \frac{\varepsilon}{3}$, and we obtain

$$|x_i(t) - z_i(t)| < \varepsilon \qquad (i = 1, 2, \ldots, n).$$

Hence the theorem: *If* $x_1(t), x_2(t), \ldots, x_n(t)$ *is a solution of* (1), *and* $z_1(t), z_2(t), \ldots, z_n(t)$ *the solution of* (14) *satisfying the same initial conditions, and if* $0 < T < \frac{1}{K} \log \left(1 + \frac{bK}{M}\right)$, *then given any positive quantity* ε, *a number* N *can be found such that if* $\nu > N$,

$$|x_i(t) - z_i(t)| < \varepsilon \qquad (0 \leqq t \leqq T;\ i = 1, 2, \ldots, n).$$

If $KT < 1$, then $C_\sigma < C$,

$$C = \frac{4\pi M(1 + KT)}{1 - KT}.$$

Hence for any ν, ι can be so chosen that if $\sigma > \iota$,

$$|x_i(t) - u_i^{(\sigma)}(t)| < \frac{1}{2\nu}, \qquad |z_i(t) - v_i^{(\sigma)}(t)| < \frac{1}{2\nu}, \qquad |u_i^{(\sigma)} - v_i^{(\sigma)}| < \frac{C}{\nu}.$$

Consequently

$$|x_i(t) - z_i(t)| < \frac{C+1}{\nu} \qquad (0 \leqq t \leqq T;\ i = 1, \ldots, n),$$

an inequality independent of ι.

2. Example

Suppose μ a parameter on the interval $0 < \mu < 1$, and assume masses μ $1 - \mu$ connected by a rigid weightless bar of unit length. Assume this system to rotate about its center of gravity in the (x, y) plane, with an angular velocity n, the origin coinciding with the center of gravity. Then if a particle of unit mass moves in space under the newtonian attraction of the first two masses, its coördinates satisfy the equations

$$\frac{dx}{dt} = x', \qquad \frac{dx'}{dt} = \frac{\partial U}{\partial x},$$

$$\frac{dy}{dt} = y', \qquad \frac{dy'}{dt} = \frac{\partial U}{\partial y},$$

$$\frac{dz}{dt} = z', \qquad \frac{dz'}{dt} = \frac{\partial U}{\partial z},$$

(18)

$$U = \frac{1-\mu}{r_1} + \frac{\mu}{r_2},$$

$$x_1 = \mu \cos nt, \qquad y_1 = \mu \sin nt,$$

$$x_2 = (1 - \mu) \cos nt, \qquad y_2 = -(1 - \mu) \sin nt,$$

$$r_1^2 = (x - x_1)^2 + (y - y_1)^2 + z^2,$$

$$r_2^2 = (x - x_2)^2 + (y - y_2) + z^2.$$

If $P_0(x_0, y_0, z_0)$ is such that $r_1^0 > 1$, $r_2^0 > 1$, or if $z_0 \neq 0$, then a certain neighborhood of P_0 can be found within which the first and second partial derivatives of U, with respect to x, y, z, are continuous and their absolute values have upper bounds independent of n. Within this neighborhood equations (18) are of the form (1), and the theorem of § 1 can be applied. The motion approaches that defined by the equations

$$(19)\qquad \begin{aligned} \frac{d^2\bar{x}}{dt^2} &= \frac{\partial \bar{U}}{\partial \bar{x}}, \\ \frac{d^2\bar{y}}{dt^2} &= \frac{\partial \bar{U}}{\partial \bar{y}}, \qquad \bar{U} = \frac{1-\mu}{2\pi}\int_0^{2\pi}\frac{d\omega}{\bar{r}_1} + \frac{\mu}{2\pi}\int_0^{2\pi}\frac{d\omega}{\bar{r}_2}, \\ \frac{d^2\bar{z}}{dt^2} &= \frac{\partial \bar{U}}{\partial \bar{z}}, \\ \bar{x}_1 &= \mu\cos\omega, \qquad \bar{y}_1 = \mu\sin\omega, \\ \bar{x}_2 &= (1-\mu)\cos\omega, \qquad \bar{y}_2 = -(1-\mu)\sin\omega. \end{aligned}$$

The limiting motion is that of a particle moving in space under the attraction of two concentric rings, each of uniform density; the equations (19) admit the area integral

$$xy' - x'y = C,$$

in addition to the energy integral. Consequently the plane problem is integrable.

The interest of this result lies in the fact that while in the restricted problem of three bodies $n = 1$, yet the analytic discussion in many cases* is precisely the same as for n arbitrary ($\neq 0$

* For instance, Birkhoff, *The restricted problem of three bodies*, Rendiconti del Circolo Matematico di Palermo, vol. 39 (1915).

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